Received: *January 20, 2000*, Accepted: *February 4, 2000*

Softly-broken $\mathcal{N} = 4$ supersymmetry in the large-N limit

Nick Dorey

Department of Physics, University of Wales Swansea
Singleton Park, Swansea, SA2 8PP, UK
E-mail: n.dorey@swansea.ac.uk

S. Prem Kumar

Department of Physics, University of Washington, Box 351560
Seattle, Washington 98195-1560, USA
E-mail: kumar@fermi.phys.washington.edu

Abstract: We calculate the exact values of the holomorphic observables of $\mathcal{N} = 4$ supersymmetric SU(N) Yang-Mills theory deformed by mass terms which preserve $\mathcal{N} = 1$ SUSY. These include the chiral condensates in each massive vacuum of the theory as well as the central charge which determines the tension of BPS saturated domain walls interpolating between these vacua. Several unexpected features emerge in the large-N limit, including anomalous modular properties under an SL$(2, \mathbb{Z})$ duality group which acts on a complexification of the 't Hooft coupling $\lambda = g^2N/4\pi$. We discuss our results in the context of the AdS/CFT correspondence.

Keywords: 1/N Expansion, Confinement, Supersymmetry and Duality, Extended Supersymmetry.

JHEP02(2000)006

Contents

1. Introduction

The AdS/CFT correspondence [1] makes non-trivial predictions for the large-N behaviour of four-dimensional gauge theories. Although the correspondence was initially applied to the $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory at the conformal point in its moduli space, subsequent developments have led to IIB supergravity duals for gauge theories with less supersymmetry obtained by perturbing the $\mathcal{N}=4$ theory with relevant operators. These perturbations can lead to new conformal field theories in the IR [2], as well as massive theories in confining [3, 4], Coulomb [5] or Higgs phases. A general feature of each of these cases is that the supergravity approximation is valid only for large values of the 't Hooft coupling, $\lambda = g^2N/4\pi \gg 1$. To go beyond this regime, one would have to evaluate stringy corrections to the supergravity limit exactly, which is not feasible at present. On the other hand, conventional gauge theory calculations are usually only possible for $\lambda \ll 1$, or not at all in the case of the confining phase. One way of making progress around this impasse is to identify special quantities on either side of the correspondence whose dependence on λ is highly constrained by supersymmetry and can be determined exactly. The main examples of this type in the $\mathcal{N}=4$ theory are the two- and three-point correlation functions of chiral primary operators [6] which are known to be completely independent of λ. Instanton contributions to four-point functions (and higher) [7] seem to be similarly protected although the origin of λ-independence in this case is less well understood (see however [11]).

In this letter we will consider the massive deformations of $\mathcal{N}=4$ SYM which preserve $\mathcal{N}=1$ supersymmetry. The resulting theory exhibits a variety of supersymmetric ground-states in confining, Coulomb and Higgs phases. As is usual for theories with $\mathcal{N}=1$ supersymmetry, there is a special class of quantities which depend holomorphically on the parameters [8]. In particular, these include the vacuum expectation values of the lowest components of chiral superfields or 'chiral condensates' for short. In four-dimensions, the $\mathcal{N}=1$ supersymmetry algebra admits a central charge which is also a holomorphic function of the parameters [9]. The resulting Bogomol'nyi bound leads to an exact formula for the tension of BPS-saturated domain walls.[1] In recent work [10], one of the authors derived the exact superpotential of the mass-deformed $\mathcal{N} = 4$ theory. In the following we will use this and related results to determine the chiral condensates in each massive supersymmetric vacuum and the tension of BPS-saturated domain walls interpolating between each pair of vacua (at least when N is a prime number). The resulting formulae transform covariantly under the S-duality group of the $\mathcal{N} = 4$ theory. In particular, the central charge transforms with modular weight two up to permutations of the vacua.

The main application of the field theory results described above is to investigate the large-N limit of the theory in its confining phase. For $\lambda \ll 1$, the scalar fields of the $\mathcal{N} = 4$ theory (and their $\mathcal{N} = 1$ superpartners) decouple and the theory reduces to pure $\mathcal{N} = 1$ supersymmetric Yang-Mills (SYM) theory. This theory is expected to be qualitatively similar to the large-N limit of QCD, exhibiting asymptotic freedom in the UV and confinement in the IR. On the other hand, for $\lambda \gg 1$, the theory potentially has a description in terms of IIB supergravity on a background which is asymptotically $AdS_5 \times S^5$. Indeed a supergravity dual has recently been proposed by Girardello et al [4]. These authors show that the dual description reproduces several qualitative properties of the confining phase such as gluino condensation, a mass gap and confinement itself. However, as the dual geometry is singular, it is not clear whether the supergravity approximation should be quantitatively reliable, even when the 't Hooft coupling is large.

Our exact formulae interpolate between these two regimes and we discuss their implications for the dual string theory. Several surprising features emerge in the large-N limit. The chiral condensates have quasi-modular properties under an $\mathrm{SL}(2,\mathbb{Z})$ duality group which acts on a complexification of the 't Hooft coupling. Taken at face value, this corresponds to an anomalous T-duality of the dual string theory which inverts the radius of the asymptotic geometry. We use this unexpected property to determine the leading behaviour of the holomorphic quantities in the $\lambda \to \infty$ limit explicitly. We find that the vacuum structure of the large-λ theory in its confining phase is qualitatively different from that of $\mathcal{N} = 1$ SYM. In the large-N limit, the latter theory has a continuous degeneracy of vacua lying on a circle centered at the origin in the complex $\mathcal{S}$-plane (here $\mathcal{S}$ denotes the gluino condensate). For $\lambda \gg 1$, we find instead an infinite but discrete set of points in the $\mathcal{S}$-plane, all lying on a finite segment of a single line through the origin. This behaviour reflects an interesting non-analyticity of the exact formulae near the point $\lambda = \infty$, $N = \infty$ in parameter space.

If the theory does have a supergravity dual for large λ, then the resulting series expansion around $\lambda = \infty$ represents a series of stringy corrections to the supergravity limit. However, we find that the series contains exponentially supressed terms which have no obvious semiclassical interpretation as worldsheet instantons [11]. By

[1] We will not, however, address the much more complicated problem of checking that these BPS domain walls are actually present in the theory.

analogy with similar phenomena in quantum field theory, we take this as a hint that the corresponding world-sheet description may be intrinsically strongly coupled. We also discuss the large-N behaviour of the theory in its unique Higgs phase vacuum. The large-N scaling of the holomorphic quantities in this vacuum raises a puzzle. In particular, we find that the tension of a BPS-saturated domain wall interpolating between vacua in the Higgs and confinement phases scales like N^4. This is hard to interpret in the context of a dual string theory with string coupling $g_{st} \sim 1/N$. We briefly discuss a possible resolution of this puzzle.

We begin by considering $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory with gauge group $\mathrm{SU}(N)$ in four dimensions. In terms of $\mathcal{N}=1$ superfields this theory contains a gauge multiplet V as well as three chiral multiplets Φ_i, $i=1,2,3$ in the adjoint representation of $\mathrm{SU}(N)$. Soft breaking to $\mathcal{N}=1$ is accomplished by introducing masses m_i for each chiral superfield Φ_i. Including these terms the classical superpotential is,[2]

$$W = N\,\mathrm{Tr}_N\left(2\sqrt{2}\Phi_1[\Phi_2,\Phi_3] + m_1\Phi_1^2 + m_2\Phi_2^2 + m_3\Phi_3^2\right). \tag{1.1}$$

Following [13], we will now analyse vacuum structure of the theory. It is convenient to rescale the chiral superfields as $\Phi_1 = i\sqrt{m_2m_3}X/\sqrt{2}$, $\Phi_2 = i\sqrt{m_3m_1}Y/\sqrt{2}$, $\Phi_3 = -i\sqrt{m_1m_2}Z/\sqrt{2}$. Up to an overall normalization the superpotential then becomes,

$$W = N\,\mathrm{Tr}_N\left(\frac{i}{2}(X^2+Y^2+Z^2) - X[Y,Z]\right). \tag{1.2}$$

To find supersymmetric ground states, we must impose the F-term equations which come from varying W with respect to X, Y and Z. The first equation is $iX=[Y,Z]$ and the other two are generated by cyclic permutation of the three superfields. A supersymmetric vacuum is therefore described by three $N\times N$ matrices which obey the commutation relations of an $\mathrm{SU}(2)$ Lie algebra. However, we still have to impose the corresponding D-term equations and mod out by $\mathrm{SU}(N)$ gauge transformations. As usual these two steps can be combined by dividing out the action of the complexified gauge group $\mathrm{SL}(N,\mathbb{C})$ on X, Y and Z. In fact, up to an $\mathrm{SL}(N,\mathbb{C})$ gauge transformation, there is exactly one solution of the $\mathrm{SU}(2)$ commutation relations in terms of $N\times N$ matrices for each N-dimensional representation of $\mathrm{SU}(2)$ [13].

We will first consider the unique irreducible representation of dimension N. Fixing the $\mathrm{SL}(N,\mathbb{C})$ gauge symmetry, we may set X, Y and Z equal to the standard generators J_X, J_Y and J_Z of this representation. Thus we have $Z=J_Z=\mathrm{diag}(m,m-1,\ldots,1-m,-m)$ with $m=(N-1)/2$. This choice of VEVs for the scalar fields completely breaks the $\mathrm{SU}(N)$ gauge symmetry and thus we have a single

[2] The normalization of the superpotential differs from [10] in two respects. Firstly, as is standard when discussing the large-N limit, we have rescaled the fields so that each term in the classical Lagrangian scales like N. Secondly, the factor of $2\sqrt{2}$ ensures that our conventions agree with those of Seiberg and Witten [12] for gauge group $\mathrm{SU}(2)$.

JHEP02(2000)006

supersymmetric vacuum state in the Higgs phase. One may wonder whether the classical analysis leading to this conclusion is reliable. This depends on whether we can choose the parameters so that the theory in this vacuum is weakly coupled. At energies much larger than $|m_i|$, the mass terms are irrelevant and we have $\mathcal{N}=4$ supersymmetric Yang-Mills theory. In this theory the β-function vanishes exactly and the complexified gauge coupling $\tau = 4\pi i/g^2 + \theta/2\pi$ does not run. On the other hand, as the gauge symmetry is completely broken, the classical vacuum has a mass gap and thus the gauge coupling does not run for mass scales much less than $|m_i|$ either. Thus, as long as the 't Hooft coupling $\lambda = g^2N/4\pi$ is much less than one the theory is weakly coupled at all energy scales as required.

In this vacuum we may easily evaluate the classical values of the chiral condensates, $u_l = \langle \mathrm{Tr}_N\, \Phi_3^l \rangle$. We find that

$$\begin{aligned} u_{2k} &= -\frac{m_1^k m_2^k}{2^{k-1}} \sum_{l=1}^{r} l^{2k} && \text{for } N = 2r+1\,; \\ &= -\frac{m_1^k m_2^k}{2^{3k-1}} \sum_{l=1}^{r} (2l-1)^{2k} && \text{for } N = 2r\,. \end{aligned} \tag{1.3}$$

Evaluating these sums explicitly for $k=1$ we find, $u_2 = m_1 m_2 N(1-N^2)/24$ for all N. Note also that the u_{2k+1} vanish for each value of N. Expanding (1.3) in the large-N limit we obtain,

$$u_{2k} = -\frac{1}{2^{3k}(2k+1)} m_1^k m_2^k N^{2k+1} \left(1 + O\left(\frac{1}{N^2}\right)\right). \tag{1.4}$$

For later convenience we introduce the normalized condensates $\tilde{u}_{2k} = u_{2k}/N^{2k+1}$, which approach constant values in the large-N limit.

We may also anticipate the form of quantum corrections to these results. As discussed above, the theory is weakly-coupled near $\tau = i\infty$ and should therefore have a semiclassical expansion around this point. Perturbative corrections correspond to powers of $1/\mathrm{Im}(\tau)$ while instanton and anti-instanton contributions are weighted by factors of $q = \exp(2\pi i \tau)$ and $\bar{q}$ respectively. However, u_{2k} is the expectation value of the lowest component of a chiral superfield and must therefore be holomorphic in τ. Hence we expect u_{2k} to have an instanton expansion in integer powers of q with no perturbative corrections. Contributions from anti-instantons or $I\bar{I}$ pairs are also ruled out by holomorphy. As, $q \sim \exp(-2\pi N/\lambda)$, each of these instanton contributions is exponentially supressed in the large-N limit. Thus we find that the classical results (1.3) for the condensates u_{2k} in the Higgs phase vacuum, are exact at each finite order in the $1/N$ expansion. Thus (1.3) yields definite predictions for the condensates in the limit $\lambda \to \infty$ which we discuss in the context of the AdS/CFT correspondence below.

The trivial representation of SU(2) yields another classical supersymmetric vacuum. In this case we have $\Phi_i = 0$ for $i = 1, 2, 3$ and the gauge group is completely unbroken. At energy scales much less than $|m_i|$, the three hypermultiplets decouple and the low-energy theory reduces to $\mathcal{N} = 1$ SUSY Yang-Mills with gauge group SU(N). This theory has a negative β-function and the gauge coupling becomes strong at the dynamical scale $\Lambda = (m_1 m_2 m_3)^{1/3} q^{1/3N}$ leading to confinement and the generation of a mass gap. Further, $\mathcal{N} = 1$ SYM has a non-anomalous Z_{2N} global symmetry which is spontaneously broken to Z_2 by gluino condensation giving N supersymmetric vacua [14]. The theory is believed to have BPS-saturated domain walls which interpolate between each pair of vacua [9]. The chiral condensates will depend holomorphically on the complexified gauge coupling τ and, as above, this rules out any perturbative contributions to these quantities. However, an important difference between the Higgs and confinement phases is that the latter is strongly-coupled in the IR irrespective of how small the gauge coupling of the UV theory is. Hence, the form of non-perturbative contributions is not constrained by semiclassical considerations. In particular, fractional powers of q, which cannot be attributed to any finite action classical field configuration, may occur.

In summary, we have found a total of $N + 1$ massive supersymmetric vacua. The theory is in the Higgs phase in one of these and the confining phase in the remainder. Of course, we have only considered vacua corresponding to the irreducible and the trivial representations of SU(2) and there are many intermediate possibilities. It turns out that as long as N is prime, all the remaining vacua correspond to a Coulomb phase without a mass gap [13]. We will not consider these massless vacuum states here. If N has non-trivial divisors, then there are also additional confining vacua where a non-abelian proper subgroup of SU(N) remains unbroken. In fact the total number of massive vacua is equal to the sum of the divisors of N. To avoid complication we will restrict our attention to those confining vacua where the full SU(N) gauge group remains unbroken (corresponding to the trivial represenation of SU(2)). These are precisely the N confining vacua that appear when N is prime. Many of the results that we present below can however, be generalised easily to include the remaining vacua as well.

In [10], one of the authors derived the exact superpotential for the theory considered above via compactification on $R^3 \times S^1$. The results presented in [10] are independent of the radius of compactification and apply equally to the theory on R^4. For the present purposes, we will only require the value of the superpotential in each vacuum as a function of the masses m_i and the complexified gauge coupling τ. Labelling the N confining vacua with an integer $m = 0, 1, \ldots, N-1$, the superpotential in each of these vacua is given as,

$$W = \frac{N^3}{24} m_1 m_2 m_3 \left[g_m(\tau, N) + A(\tau, N) \right], \tag{1.5}$$

while in the Higgs vacuum we have,

$$W = \frac{N^3}{24} m_1 m_2 m_3 \left[h(\tau, N) + A(\tau, N) \right], \tag{1.6}$$

with,

$$g_m(\tau,N) = E_2(\tau) - \frac{1}{N}E_2\left(\frac{\tau+m}{N}\right)$$
$$h(\tau,N) = E_2(\tau) - NE_2(N\tau)\,. \tag{1.7}$$

Here $A(\tau,N)$ is an undetermined holomorphic function of τ with a weak coupling expansion of the form,

$$A(\tau,N) = \alpha_0 + \sum_{k=1}^{\infty}\alpha_k q^k\,, \tag{1.8}$$

which will be discussed further below.

In the above, $E_2(\tau)$ is the regulated second Eisenstein series [15],

$$E_2(\tau) = \frac{3}{\pi^2}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}{}' \frac{1}{(m+n\tau)^2}\,. \tag{1.9}$$

The prime on the m summation means 'omit the $m=0$ term when $n=0$'. In the weak coupling limit,[3] $\mathrm{Im}(\tau)\to\infty$, $E_2(\tau)$ can be expanded as,

$$E_2(\tau) = 1 - 24\sum_{k=1}^{\infty}c_k q^k\,, \tag{1.10}$$

with $c_k = \sum_{d|k} d$. Hence we see that the weak coupling expansion of the superpotential in the Higgs vacuum contains only integral powers of q as anticipated above. In contrast, the expansion of the superpotential in each of the confining vacua is a power series in $q^{1/N}$. These contributions formally correspond to fractional instantons or 'merons'. Note that we have $q^{1/N}\sim\exp(-2\pi/\lambda)$ in the 't Hooft limit. This means that unlike the effects of conventional instantons which are exponentially supressed at large-N, fractional instanton contributions are independent of N in this limit. Of course, there are no corresponding solutions of the classical field equations with fractional toplogical charge for a gauge theory on R^4. The presence of fractional powers of q simply reflects the fact that the theory in the confining phase is intrinsically strongly coupled and thus semiclassical reasoning is invalid. Interestingly however, for the theory compactified on $R^3\times S^1$, a semiclassical interpretation of merons does emerge (at least when the radius of S^1 is small): they are precisely magnetic monopoles which contribute as three-dimensional instantons after compactification [16, 17] (see also [18] and references therein).

The superpotential determines the value of the condensates $u_2 = \langle \mathrm{Tr}_N\,\Phi_3^2\rangle$ and $\mathcal{S} = \langle \mathrm{Tr}_N\,W_\alpha W^\alpha\rangle$ via the standard relations,

$$u_2 = \frac{1}{N}\frac{\partial W}{\partial m_3}\,, \qquad \mathcal{S} = -8\pi i\frac{\partial W}{\partial \tau}\,. \tag{1.11}$$

[3]In the following, we will use the term 'weak-coupling' to describe the expansion of our exact results for large $\mathrm{Im}(\tau)$ even though, as explained above, the theory in the confining phase is not weakly coupled in this regime.

Using the methods of [10], one may also calculate the exact values of the chiral condensates u_{2k} with $k > 1$ in each vacuum of the theory. Some explicit results along these lines are given in the appendix.

From the first relation in (1.11), we see that the unknown function $A(\tau, N)$ results in a nonperturbative ambiguity in the condensate u_2. As we discuss below, this coincides with the ambiguity [19] in the definition of the corresponding quantity in $\mathcal{N} = 2$ SYM with one adjoint hypermultiplet. The first coefficient, α_0, may be fixed by comparison with the classical analysis given above. In particular, we require that u_2 vanishes in the weak-coupling limit $\mathrm{Im}(\tau) \to \infty$ in each confining vacuum. This requires $\alpha_0 = (1 - N)/N$. If we now substitute this value into the corresponding formula for the Higgs branch vacuum we find $u_2 \to m_1 m_2 N(1-N^2)/24$ as $\mathrm{Im}(\tau) \to \infty$ in precise agreement with eq. (1.3). The rest of the coefficients α_k, $k > 0$, remain unknown except in the case of gauge group SU(2) where α_1 was determined in [19] by an explicit instanton calculation.

The above formulae have interesting properties under the S-duality group of $\mathcal{N} = 4$ SUSY Yang-Mills. Specifically for instance, when N is prime, under the electric-magnetic duality transformation $S : \tau \to -1/\tau$ we have,

$$\begin{aligned} g_0\left(-\frac{1}{\tau}, N\right) &= \tau^2 h(\tau, N)\,, \qquad g_m\left(-\frac{1}{\tau}, N\right) = \tau^2 g_p(\tau, N)\,, \\ h\left(-\frac{1}{\tau}, N\right) &= \tau^2 g_0(\tau, N)\,, \end{aligned} \tag{1.12}$$

where the second equality applies for $m = 1, \ldots, N-1$, with $p(m, N)$ denoting the least positive integer satisfying $pm = -1 \bmod N$. Under the other generator of the modular group, $T : \tau \to \tau + 1$ we have,

$$\begin{aligned} g_m(\tau + 1, N) &= g_{m+1}(\tau, N)\,, \qquad g_{N-1}(\tau + 1, N) = g_0(\tau, N)\,, \\ h(\tau + 1, N) &= h(\tau, N)\,. \end{aligned} \tag{1.13}$$

Hence the set of functions $\{g_0, g_1, \ldots, g_{N-1}, h\}$ transform as modular forms of weight two modulo permutations. In the SU(2) case, the three functions in question are just the well-known quasi-modular forms, $e_1(\tau)$, $e_2(\tau)$ and $e_3(\tau)$. The presence of the unknown function $A(\tau, N)$ implies that these modular properties are not inherited by the superpotential itself. In order to recover modular covariant results we have to work with the shifted superpotential $\tilde{W} = W - N^3 m_1 m_2 m_3 A(\tau, N)/24$. Fortunately, as we will see below, this shift does not affect the domain wall tensions of the theory as these only depend on the differences of the value of the superpotential in different vacua.

The fact that $\mathrm{SL}(2, \mathbb{Z})$ transformations permute the different vacua has a nice explanation in terms of the corresponding theory with $\mathcal{N} = 2$ supersymmetry which is obtained by setting $m_1 = m_2 = M$ and $m_3 = 0$ [12, 20]. In this case the theory has a Coulomb branch parameterized by the moduli u_l introduced above, which is determined by a hyperelliptic curve. On special singular submanifolds of the Coulomb branch the curve degenerates producing massless BPS states. If we now reintroduce a non-zero value for m_3, thereby breaking $\mathcal{N}=2$ supersymmetry down to an $\mathcal{N}=1$ subalgebra, the massless BPS states at each singular point condense giving a massive supersymmetric vacuum. The values of the condensates u_l in each vacuum of the $\mathcal{N}=1$ theory are determined by the location of the corresponding singular point on the Coulomb branch of the $\mathcal{N}=2$ theory. The natural action of the S-duality group on the BPS spectrum is then inherited by the singular points on the Coulomb branch and thus by the corresponding $\mathcal{N}=1$ vacua. In the case of gauge group SU(2), the ambiguity arising from the unknown function $A(\tau,N)$, corresponds to the mismatch between the parameter $\tilde{u}$ appearing in the Seiberg-Witten curve [12], which transforms with modular weight two, and the physical quantity $u=\langle\mathrm{Tr}_2\,\Phi^2\rangle$ which is not modular covariant [19].

In general the theory may contain BPS-saturated domain walls which interpolate between each pair of supersymmetric vacua. To determine whether BPS-saturated domain walls exist, we would need more information about the theory beyond the holomorphic quantities calculated in this paper. However, if these states are present, the exact formula for their tension is $T=|\Delta W|$ where the central charge ΔW is the difference between the value of the superpotential in each of the two vacua [9]. Thus we may find a BPS saturated domain wall interpolating between the Higgs vacuum and each confining vacuum with,

$$\Delta W_m = \frac{N^4}{24}m_1m_2m_3\left[E_2(N\tau)-\frac{1}{N^2}E_2\left(\frac{\tau+m}{N}\right)\right] \tag{1.14}$$

and between each pair of confining vacua with,

$$\Delta W_{m,n} = \frac{N^2}{24}m_1m_2m_3\left[E_2\left(\frac{\tau+m}{N}\right)-E_2\left(\frac{\tau+n}{N}\right)\right]. \tag{1.15}$$

The fact that these expressions scale differently in the large-N limit will be significant in what follows. As $\mathrm{SL}(2,\mathbb{Z})$ transformations permute the supersymmetric vacua, they also permute the domain walls which interpolate between each pair of vacua. In particular, the domain wall tensions $T=|\Delta W|$ transform with (non-holomorphic) modular weight $(1,1)$ up to permutations. Given the existence of one type of BPS-domain wall in the theory, this modular behaviour suggests that all the others should be present.

The theory in each of the confining vacua flows to pure $\mathcal{N}=1$ supersymmetric Yang-Mills theory in the IR. More precisely, to decouple the extra degrees of freedom of the $\mathcal{N}=4$ theory we must must ensure that the dynamical scale $\Lambda=(m_1m_2m_3)^{1/3}q^{1/3N}$ is much less than each of the chiral multiplet masses. This will be the case if the 't Hooft coupling $\lambda=g^2N/4\pi$ is much less than one. In this regime it is easy to check that our formulae reproduce the standard results for $\mathcal{N}=1$

SUSY Yang-Mills. In particular, the gluino condensate in each of the confining vacua is [14],

$$\begin{aligned}\mathcal{S}(m) &= 16\pi^2 N m_1 m_2 m_3 \exp\left(-\frac{8\pi^2}{g^2 N}\right)\exp\left(i\frac{2\pi m}{N}+i\frac{\theta}{N}\right)\\ &= 16\pi^2 N\Lambda^3 \exp\left(\frac{2\pi i m}{N}\right),\end{aligned} \tag{1.16}$$

for $m = 0, 1, \ldots, N-1$. The leading correction to the above formula is of order $\exp(-16\pi^2/g^2N)$ which is small provided $\lambda \ll 1$. Note that the θ parameter has been absorbed in the definition of the dynamical scale Λ reflecting the presence of an anomalous $\mathrm{U}(1)_R$ symmetry which appears in the decoupling limit. The confining vacua lie at the vertices of a regular N-gon in the complex $\mathcal{S}$-plane. This reflects the spontaneous breaking of a non-anomalous Z_{2N} subgroup of $\mathrm{U}(1)_R$ down to Z_2. Importantly, neither $\mathrm{U}(1)_R$ or its anomaly-free subgroup are exact symmetries of the mass-deformed $\mathcal{N}=4$ theory. We also find the standard formula for the tension of BPS saturated domain walls interpolating between pairs of confining vacua [9],

$$T_{m,n} = |\Delta W_{m,n}| = 16\pi^2 N^2 \Lambda^3 \left|1 - e^{\frac{2\pi i}{N}(m-n)}\right|. \tag{1.17}$$

Note that the Higgs phase vacuum of the mass-deformed $\mathcal{N}=4$ theory effectively decouples in this limit. In particular the ratio of the tension of a domain wall interpolating between the Higgs and confining phases to that of one interpolating between two confining vacua diverges as $\lambda \to 0$.

In [21], Witten made the interesting proposal that the domain wall of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory can act as Dirichlet brane for the confining string. The modular properties of the supersymmetric vacua and the domain walls which may interpolate between them described above suggests a generalization of this phenomenon which may occur in the present model. As explained above, each supersymmetric vacuum is associated with the condensation of a set of BPS states of the $\mathcal{N}=2$ theory of [20]. This in turn leads to the confinement of other BPS states of the $\mathcal{N}=2$ theory. The simplest example is that condensation of electrically charged elementary fields in the Higgs phase vacuum leads to the confinement of magnetic monopoles and vice versa for the confining vacuum with $m=0$. The confinement of each kind of charge requires a string which carries the corresponding flux. This suggests that we should find an $\mathrm{SL}(2,\mathbb{Z})$ multiplet of confining strings each of which can end on one kind of domain wall. The situation is similar to the properties of strings and fivebranes in Type IIB string theory. There one finds an $\mathrm{SL}(2,\mathbb{Z})$ multiplet of strings and of the fivebranes on which they may terminate. Each fivebrane acts as a Dirichlet brane for one type of string. It is tempting to believe that this similarity with the IIB theory is more than a coincidence and can be explained in terms of the AdS/CFT correspondence.

In the rest of the paper we will discuss the physics of the 't Hooft limit $N \to \infty, g^2 \to 0$ with $\lambda = g^2N/4\pi$ held fixed. On general grounds [22], we expect the large-

N gauge theory to have a dual description as a string theory with string coupling g_{st} proportional to $1/N$. As the theory in the confining phase reduces to $\mathcal{N} = 1$ supersymmetric Yang-Mills for $\lambda \ll 1$, we will begin by reviewing the large-N limit of this theory in as discussed in [21]. First note that the Λ-parameter defined above depends only on the 't Hooft coupling and not on N itself. Thus the dynamical mass scale of the theory is held fixed as $N \to \infty$. The theory is believed to contain an infinite tower colour singlet excitations or 'glueballs' with masses of order N^0 in units of Λ. One may also argue that each term in the effective action for these degrees of freedom scales like N^2, reflecting the fact that the glueballs are stable and weakly interacting in the large-N limit. This is consistent with the idea that the large-N gauge theory is equivalent to a weakly coupled string theory. The leading order terms in the effective action are then identified as a genus zero contribution of the string. As above, the confining vacua lie at the vertices of a regular N-gon in the complex $\mathcal{S}$-plane. As $N \to \infty$, the N-gon approaches a circle and we have a continuous degeneracy of vacua. Note, however, that this does *not* imply that a massless goldstone mode appears in the large-N limit.[4] One may also try to identify the string theory states which correspond to the BPS saturated domain walls of the theory. Naively the tension $T_{m,n}$ of a domain wall given in (1.17) above scales like $1/g_{st}^2$ which is typical of a finite energy classical field configuration such as a soliton. However, this is only true if $m - n \sim N$. For domain walls interpolating between adjacent vacua (or any pair of vacua with $m - n \sim N^0$) we find instead $T_{m,m+1} \sim 1/g_{st}$. This scaling is typical for a Dirichlet brane rather than a soliton. This observation motivates Witten's proposal that the domain walls are D-branes for the confining string. The above features of $\mathcal{N} = 1$ SYM can also be understood by realizing this theory on the worldvolume of an M-theory fivebrane [21].

As the $\mathcal{N} = 1$ theory discussed in this paper can be realized as a relevant deformation of $\mathcal{N} = 4$ SUSY Yang-Mills, the dual string theory should be determined by the AdS/CFT correspondence. In particular, the large-N gauge theory should have a dual description in terms of IIB string theory, with string coupling $g_{st} \sim 1/N$, on a spacetime manifold which is asymptotically $AdS_5 \times S^5$. The 't Hooft coupling of the gauge theory corresponds to the ratio $L^4/4\pi\alpha'^2$ on the IIB side, where L is the common radius of the AdS_5 and S^5 factors of the asymptotic geometry and $\sqrt{\alpha'}$ is the string length-scale. In the case of the undeformed $\mathcal{N} = 4$ theory, the IIB background is exactly $AdS_5 \times S^5$ and IIB string theory can be approximated by IIB supergravity as long as $\lambda \gg 1$. However, for the mass-deformed theory, the relevant IIB geometry may be singular, in which case the supergravity approximation is not valid even for large λ [4]. In principle, our field theory results yield predictions for the dual string theory. However, for a meaningful comparison we must identify the observables on the IIB side which correspond to the holomorphic gauge theory

[4]The explanation of this point is related to the fact that the U(1) problem of the η mass in QCD can be resolved within the context of the $1/N$ expansion [26].

quantities calculated in this paper. As discussed further below, chiral condensates correspond to certain moduli of the IIB geometry [23, 24]. In addition, domain walls interpolating between vacua should correspond to IIB brane configurations which preserve two supercharges although this has not been investigated previously. The double expansion of the holomorphic quantities in inverse powers[5] of N and λ then provides information about string-loop and worldsheet effects respectively. Obviously, most of these predictions would be very hard to test given our current ignorance about string theory in Ramond-Ramond (RR) backgrounds. However, the main point of the analysis is to obtain explicit results for the holomorphic observables in the limit $\lambda \to \infty$ and investigate the implications for a dual description in terms of IIB supergravity like that proposed in [4].

We will now consider the large-N behaviour of the mass-deformed $\mathcal{N}=4$ theory in the N confining vacua labelled by the integer $m = 0, 1, \ldots, N-1$. Note that m always appears in the formulae via the combination $\kappa = (m + \theta/2\pi)/N$. As we vary m and θ, κ can take any value in the interval $[0, 1]$. Initially we will think of κ as a continuous variable of order N^0. In fact κ itself only enters in a complex combination with the 't Hooft coupling,

$$\rho = \frac{\tau + m}{N} = \frac{i}{\lambda} + \kappa\,. \tag{1.18}$$

We begin by implementing the 't Hooft limit in the standard way, taking the $N \to \infty$ limit with λ (and thus ρ) held fixed. The gluino condensate in each confining vacuum is given by,

$$\mathcal{S}(m) = \frac{N\pi i}{3} m_1 m_2 m_3 \left[E_2'(\rho) + O\left(\exp\left(\frac{-2\pi N}{\lambda}\right)\right)\right]. \tag{1.19}$$

Note that the ambiguity in the definition of the condensate corresponding to the unknown function $A(\tau, N)$ is exponentially supressed in the large-N limit.

Interestingly, the leading order term has quasi-modular properties in the complexfied 't Hooft coupling $\rho = i/\lambda + \kappa$. In particular, the second Eisenstein series $E_2(\rho)$ is invariant under $\rho \to \rho + 1$ but has the following anomalous transformation law under the other generator of $\mathrm{SL}(2, \mathbb{Z})$,

$$E_2(\rho) = \frac{1}{\rho^2} E_2\left(-\frac{1}{\rho}\right) - \frac{6}{\pi i \rho}\,. \tag{1.20}$$

The violation of modular invariance is mild in the sense that it can be recovered by modifying $E_2(\rho)$ by a non-holomorphic correction [25]. The behaviour of the gluino condensate under a general modular transformation $\rho \to \tilde{\rho} = (a\rho + b)/(c\rho + d)$, with integers a,b,c,d satisfying $ad - bc = 1$, is governed by the formula,

$$E_2'(\rho) = \frac{1}{(c\rho + d)^4} E_2'(\tilde{\rho}) - \frac{2c}{(c\rho + d)^3} E_2(\tilde{\rho}) + \frac{6}{\pi i} \frac{c^2}{(c\rho + d)^2}\,. \tag{1.21}$$

[5] The expansions also contain non-perturbative terms in each of these parameters.

This quasi-modular behaviour in ρ originates in the modular properties of the function $g_m(\tau, N)$ appearing in the exact superpotential (1.5) under S-duality. In fact the anomalous term reflects the fact that S-duality transformations do not commute with the 't Hooft limit. However, although it has its origin in S-duality on the gauge theory side, the modular group acting on ρ does not correspond to the usual S-duality group of IIB string theory. If we set $\kappa = 0$ (ie set $\theta = 0$ and focus on the confining vacuum with $m = 0$), then the transformation $\rho \to -1/\rho$ simply inverts the 't Hooft coupling $\lambda \to 1/\lambda$. As above the 't Hooft coupling λ is identified with the ratio $L^4/4\pi\alpha'^2$ on the IIB side. Therefore, in terms of string theory parameters, the corresponding transformation is an anomalous T-duality which inverts the radius of the asymptotic geometry!

The anomalous transformation law (1.21) allows us to obtain the expansion of the gluino condensate in the supergravity limit[6] $\lambda \to \infty$. Explicitly (with $\kappa = 0$) we have,

$$E_2'(\rho) = E_2'\left(\frac{i}{\lambda}\right) = -2i\lambda^3 + \frac{6i}{\pi}\lambda^2 - 48i\lambda^3 \sum_{k=1}^{\infty}(\pi k\lambda - 1)c_k \exp(-2\pi k\lambda)\,, \qquad (1.22)$$

with $c_k = \sum_{d|k} d$ as above. With the standard identification of parameters given above, (1.22) represents a series of stringy corrections to the supergravity limit, with expansion parameter $\alpha'^2/L^4 = 1/\lambda$. In a more conventional string theory compactification without background RR fields, one would hope to identify exponentially suppressed terms like those appearing in (1.22) as the saddle-point contribution of an instanton obtained by wrapping an appropriate BPS state around a non-contractible cycle in spacetime. In fact, as the string coupling does not enter explicitly, the most natural possibility would be worldsheet instantons coming from wrapping of the IIB string itself [11]. This is not promising in the present context for a number of reasons. A worldsheet instanton would typically have an action proportional to $\sqrt{\lambda} \sim L^2/\alpha'$ rather than λ itself. In fact, on general grounds, one would also expect the subleading terms in (1.22) to be down by powers of $1/\sqrt{\lambda}$ rather than $1/\lambda$. In any case there are no topologically non-trivial cycles of the right dimension in $AdS_5 \times S^5$, nor are any apparant in the deformation of this geometry considered in [4]. Thus there is no obvious semiclassical interpretation for the exponentially suppressed terms appearing in (1.22). Because very little is known about worldsheet effects for string theory in RR backgrounds, it is hard to draw any definite conclusion from this. However it is perhaps worth making a comparison with a similar phenomena we met in a different context above: the occurence of fractional instanton contributions in the confining vacua of the four dimensional gauge theory. By analogy with that case, our result may simply suggest that the relevant world-sheet description is intrinsically strongly coupled.

[6]For convenience we will refer to the limit $\lambda \to \infty$ as the supergravity limit even though a supergravity description may not be valid there.

JHEP02(2000)006

In our discussion of $\mathcal{N} = 1$ SUSY Yang-Mills theory at large-N we found a continuous degeneracy of confining vacua lying on a circle centred at the origin of the complex $\mathcal{S}$-plane. This analysis applies to the confining vacua of the mass-deformed $\mathcal{N} = 4$ theory for $\lambda \ll 1$. From the point of view of the AdS/CFT correspondence, an obvious question is what happens to this picture for $\lambda \gg 1$. The large-λ expansion of the gluino condensate given in eq. (1.22) is specific to the confining vacuum with $m = 0$ at $\theta = 0$. The generalization of this formula to the remaining vacua is surprisingly subtle.[7] One sensible way to study the large-N limit in the confining vacua with $m \neq 0$ is to consider various sub-sequences $\{m(N)\}$ such that $\lim_{N\to\infty} m(N)/N$ converges to some rational number p/q.[8] Here p and q are mutually prime. It is then convenient to perform a modular transformation of the form,

$$\rho \to \tilde{\rho} = \frac{a\rho + b}{q\rho - p}, \tag{1.23}$$

using (1.21), where the modular condition is satisfied by choosing a and b such that $ap + bq = -1$. In the large-λ limit, the leading behaviour of the gluino condensate in the vacuum labelled by a rational number p/q is,

$$\mathcal{S} = \frac{2\pi}{3} N m_1 m_2 m_3 \frac{\lambda^3}{q^2}. \tag{1.24}$$

Note that for given q, the gluino condensate has the same leading behaviour for every p less than q such that p and q are coprime. Thus, in the large-N limit, we find an infinite tower of vacua located at the points,

$$\mathcal{S}(j) = \frac{2\pi}{3} N m_1 m_2 m_3 \frac{\lambda^3}{j^2} \tag{1.25}$$

in the complex $\mathcal{S}$-plane where j runs over the positive integers. Further, the above formula is only valid for $\theta = 0$. Changing θ continuously from 0 to 2π induces a complicated rearrangement of the vacua. Although the vacua have a point of accumulation at $\mathcal{S} = 0$ there is no trace of the continuous vacuum degeneracy which appears in the large-N limit at small λ. Note also that the discrete vacuum degeneracy for large λ is associated with the magnitude of the condensate only: the phase of $\mathcal{S}$ is the same in each vacuum. In contrast, for small λ, the magnitude of the condensate is the same in each vacuum because of the spontaneously broken Z_{2N} symmetry which appears in this limit.

[7] For any finite λ, the vacua of the large-N theory will fill a closed curve in the $\mathcal{S}$-plane, parameterized by $\kappa \in [0, 1]$, which generalizes the circle of the $\mathcal{N} = 1$ SYM case. However as λ increases the curve becomes increasingly irregular and it does not approach a well-defined limit as $\lambda \to \infty$.

[8] Other ways of taking the large-N, large-λ limit — e.g. taking λ to infinity keeping N fixed also leaves us with the rational points only with the gluino condensate having the same leading behaviour as obtained below.

A concrete application of the field theory results described above is to test the dual description in terms of IIB supergravity proposed in [4]. In particular, the chiral condensates in a given vacuum correspond to the amplitudes of certain normalizable zero modes in the expansion of the asymptotic supergravity fields around the $AdS_5 \times S^5$ background [23, 24]. Indeed, the authors of [4] used this correspondence to demonstrate the existence of a non-zero gluino condensate starting from their supergravity solution. For the configuration considered in [4], the magnitude of the gluino condensate is essentially a free parameter which arises as a constant of integration when solving the five-dimensional field equations. This corresponds to a continuous degeneracy of supersymmetric vacua on the field theory side of the correspondence which disagrees with the results presented above. Presumably this means that stringy effects remain important even at large 't Hooft coupling and must have the effect of lifting the continuous vacuum degeneracy. However, the supergravity description does reproduce one qualitative feature of the field theory results.[9] Specifically, the vacuum degeneracy implied by the supergravity solution of [4] is associated with the magnitude of the condensate rather than its phase in agreement with the large-λ formula (1.25).

So far we have only considered the large-N limit of the theory in its confining phase. However, in the spirit of the AdS/CFT correspondence, we might expect that each vacuum state of the theory has a dual description in terms of weakly-coupled IIB string theory on a manifold which is asymptotically $AdS_5 \times S^5$. Similarly, one expects that the BPS-saturated domain wall interpolating between each pair of vacua should have a counterpart in the IIB Hilbert space. However, if we try to extend the above discussion to the unique vacuum of the theory in the Higgs phase, we immediately confront a problem. According to equation (1.14), the exact tension of the domain wall which interpolates between the Higgs and confinement phases grows like $N^4 \sim 1/g_{st}^4$ which is unacceptable in a weakly-coupled string theory.[10] At this point we may simply decide that our assumptions were too strong and that the theory in the Higgs phase vacuum cannot be described in terms of a IIB background which is asymptotically $AdS_5 \times S^5$. However we will also explore the consequences of another more speculative resolution. Firstly, note that the appropriate normalization of the domain wall tension for a comparison with the tension of a BPS state on the IIB side of the AdS/CFT correspondence is not obvious. In particular it is possible that extra factors of $g_{st} \sim 1/N$ could appear. It is therefore possible that the N^4 scaling of the exact superpotential in the Higgs phase vacuum actually corresponds to the leading order in string perturbation theory on the IIB side. Of course this means that we should interpret the superpotential in the confining phase, which scales like N^2, as a genus one effect. With this interpretation, there is a continuous degeneracy of confin-

[9] The authors would like to thank Alberto Zaffaroni for this observation.

[10] Of course one way to avoid this conclusion is simply to suppose that the offending BPS state does not exist.

ing vacua at string tree-level which is lifted by a one-loop effect in string perturbation theory. Clearly this clashes with the standard discussion of the large-N limit of $\mathcal{N}=1$ SYM as reviewed above. In particular the domain walls interpolating between confining vacua with $m-n\sim N$, would now correspond to states with tension of order g_{st}^0 on the IIB side which could not be interpreted as classical solitons. The interpretation of domain walls interpolating between adjacent vacua as D-branes would also be spoilt.

Finally we will consider the large-λ limit of the chiral condensates in the Higgs phase vacuum. The classical formulae (1.3) for the u_{2k} are exact to all finite orders in the $1/N$ expansion (ie the only corrections are non-perturbative in $1/N$). In particular, u_{2k} scales like N^{2k+1} in the large-N limit with a coefficient which does not depend on λ. As for the gluino condensate, the condensates u_{2k} are associated with the amplitudes of certain normalizable modes in the expansion of the IIB background around $AdS_5\times S^5$ [23, 24]. The fact that the condensates grow rapidly with N suggests that an expansion around $AdS_5\times S^5$ may not make sense. This seems to favour the more conservative resolution of the puzzle discussed above, namely that the theory in the Higgs vacuum does not have a sensible large-N limit. However, once again, this conclusion depends on our assumptions about the correct normalization of the quantities on either side of the correspondence. In particular, in the above, we defined the normalized condensates $\tilde{u}_{2k}$ which tend to the constant values $-m_1^k m_2^k/2^{3k}(2k+1)$ in the large-N limit. An alternative interpretation is that the $\tilde{u}_{2k}$ rather than the u_{2k} should be identified with the corresponding modes of the supergravity fields in an expansion around $AdS_5\times S^5$. With this new interpretation, our results suggest that this ground state should have a dual description in terms of a classical solution of the supergravity equations. Unlike the proposed supergravity dual for the confining vacuum, this solution should be an isolated one (in otherwords there should be no vacuum degeneracy at leading order in $1/N$). The corresponding configuration should involve non-zero background values for the supergravity fields which couple to the chiral operators $\tilde{u}_{2k}$ of the gauge theory. According to the ideas of [5, 23, 24], the relevant configuration should correspond to a non-trivial distribution of D3 branes in ten dimensions. However there will also be background values for the NS-NS antisymmetric tensor field which couple to the bare masses of the fermions in each chiral multiplet. The presence of these background fields breaks $\mathcal{N}=4$ supersymmetry down to an $\mathcal{N}=1$ subalgebra (in four-dimensional conventions) just as the chiral multiplet masses do on the Yang-Mills side of the correspondence. Our result for the condensates $\tilde{u}_{2k}$, should then translate into a quantitative prediction for the holomorphic moments of the D3-brane distribution.

Acknowledgments

The authors would like to thank Massimo Porrati and Alberto Zaffaroni for commenting on a first draft of this paper. N.D. also would like to thank Michael Green and Misha Shifman for useful discussions. N.D. acknowledges the support of T.M.R. network grant FMRX-CT96-0012 and of a PPARC advanced research fellowship. S.P.K. would like to acknowledge the support of DOE grant DE-FG03-96ER40956.

A. Some exact results for the condensates u_{2k}

In this appendix, we give some exact results for the condensates u_{2k} for $k > 1$, a more detailed discussion of these results will appear elsewhere [27]. In general these quantities will suffer from ambiguities like the one discussed in the text for u_2. However, we will suppress this fact in the following.

In [10], one of the authors analysed the theory considered here via compactification on $R^3 \times S^1$. The main result was an exact expression for the superpotential on the Coulomb branch of the three-dimensional effective theory, which coincides with the complexified potential of the elliptic Calogero-Moser system. This result, which can be deduced from holomorphy combined with various semiclassical considerations, can also be derived from the general connection [28, 29, 20] between integrable systems and supersymmetric gauge theories. A novel feature discussed in [10], is that each equilibrium configuration of the integrable system corresponds to a supersymmetric vacuum of the $\mathcal{N} = 1$ theory. The condensates $u_l = \langle \mathrm{Tr}\, \Phi_3^l \rangle$ in each vacuum are then identified with the constants of motion of the Calogero-Moser integrable system in a given equilibrium configuration. The exact values for the chiral condensates in the massive vacua are implicitly given by,

$$
\begin{aligned}
C_{2k}(u) &= \frac{(-1)^{2k} m_1^k m_2^k}{(2k)!!} \sum_{a_1 \neq a_2 \neq a_3 \cdots \neq a_{2k}} \mathcal{P}(X_{a_1} - X_{a_2}) \mathcal{P}(X_{a_3} - X_{a_4}) \cdots \mathcal{P}(X_{a_{2k-1}} - X_{a_{2k}})\,, \\
C_{2k+1}(u) &= 0\,, \qquad \qquad \text{(A.1)}
\end{aligned}
$$

where $\mathcal{P}$ is the Weierstrass function and the C_k are Schur polynomials in u_k which are the coefficients of t^{-k} in $\sum_{k=0} C_k(u) t^{-k} = \exp[-\sum_{s=0}^{\infty} u_s / t^{-s}]$. In the Higgs vacuum we have $X^a = 2\pi i a/N$, $(a = 1, \ldots, N-1)$ while in the confining vacuum with $m = 0$ discussed in the text we have $X^a = 2\pi i \tau a/N$, $(a = 1, \ldots, N-1)$.

It is then easy to see that in the 't Hooft large-N limit $u_{2k} \sim m_2^k m_1^k N^{2k+1}$ in the Higgs vacuum which coincides with our earlier weak-coupling analysis (1.3). On the other hand in the confining vacuum we find that $u_{2k} \sim m_1^k m_2^k N^{k+1}$. An explicit example is provided by u_2 which by eq. (A.1) is equal to $\frac{m_1 m_2}{2} \sum_{a_1 \neq a_2} \mathcal{P}(X_{a_1} - X_{a_2})$. This sum can be evaluated explicitly to obtain the exact results for u_2 given in the text. If we consider the appropriately normalised condensates $\tilde{u}_{2k} = u_{2k}/N^{2k+1}$, then in the Higgs vacuum they approach their semiclassical values, while in the confining vacua they vanish in the large-N limit.

References

[1] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200];
S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109];
E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150];
O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *large-N field theories, string theory and gravity*, hep-th/9905111.

[2] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, *Novel local CFT and exact results on perturbations of $N = 4$ super Yang-Mills from AdS dynamics*, *J. High Energy Phys.* **12** (1998) 022 [hep-th/9810126];
J. Distler and F. Zamora, *Non-supersymmetric conformal field theories from stable anti-de Sitter spaces*, *Adv. Theor. Math. Phys.* **2** (1999) 1405 [hep-th/9810206];
A. Khavaev, K. Pilch and N.P. Warner, *New vacua of gauged $N = 8$ supergravity in five dimensions*, hep-th/9812035.

[3] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, *Confinement and condensates without fine tuning in supergravity duals of gauge theories*, *J. High Energy Phys.* **05** (1999) 026 [hep-th/9903026].

[4] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, *The supergravity dual of $N = 1$ super Yang-Mills theory*, hep-th/9909047.

[5] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, *Continuous distributions of D3-branes and gauged supergravity*, hep-th/9906194;
A. Brandhuber and K. Sfetsos, *Wilson loops from multicentre and rotating branes, mass gaps and phase structure in gauge theories*, hep-th/9906201;
I. Chepelev and R. Roiban, *A note on correlation functions in AdS_5/SYM_4 correspondence on the Coulomb branch*, *Phys. Lett.* **B 462** (1999) 74 [hep-th/9906224].

[6] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, *Three-point functions of chiral operators in $D = 4$, $N = 4$ SYM at large-N*, *Adv. Theor. Math. Phys.* **2** (1998) 697 [hep-th/9806074];
E. D'Hoker, D.Z. Freedman and W. Skiba, *Field theory tests for correlators in the AdS/CFT correspondence*, *Phys. Rev.* **D 59** (1999) 045008 [hep-th/9807098].

[7] N. Dorey, T.J. Hollowood, V.V. Khoze, M.P. Mattis and S. Vandoren, *Multi-instanton calculus and the AdS/CFT correspondence in $N = 4$ superconformal field theory*, *Nucl. Phys.* **B 552** (1999) 88 [hep-th/9901128].

[8] N. Seiberg, *Naturalness versus supersymmetric nonrenormalization theorems*, *Phys. Lett.* **B 318** (1993) 469 [hep-ph/9309335].

[9] G. Dvali and M. Shifman, *Domain walls in strongly coupled theories*, *Phys. Lett.* **B 396** (1997) 64 [hep-th/9612128];

A. Kovner, M. Shifman and A. Smilga, *Domain walls in supersymmetric Yang-Mills theories*, *Phys. Rev.* **D 56** (1997) 7978 [hep-th/9706089];
B. Chibisov and M. Shifman, *BPS-saturated walls in supersymmetric theories*, *Phys. Rev.* **D 56** (1997) 7990 [hep-th/9706141].

[10] N. Dorey, *An elliptic superpotential for softly broken $N = 4$ supersymmetric Yang-Mills theory*, *J. High Energy Phys.* **07** (1999) 021 [hep-th/9906011].

[11] R. Gopakumar and M.B. Green, *Instantons and non-renormalisation in AdS/CFT*, *J. High Energy Phys.* **12** (1999) 015 [hep-th/9908020].

[12] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099].

[13] C. Vafa and E. Witten, *A strong coupling test of s duality*, *Nucl. Phys.* **B 431** (1994) 3 [hep-th/9408074].

[14] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Supersymmetric instanton calculus: gauge theories with matter*, *Nucl. Phys.* **B 260** (1985) 157;
D. Amati, K. Konishi, Y. Meurice, G.C. Rossi and G. Veneziano, *Nonperturbative aspects in supersymmetric gauge theories*, *Phys. Rep.* **162** (1988) 169.

[15] N. Koblitz, *Introduction to elliptic curves and modular forms*, Springer-Verlag, New York 1984.

[16] K. Lee and P. Yi, *Monopoles and instantons on partially compactified D-branes*, *Phys. Rev.* **D 56** (1997) 3711 [hep-th/9702107].

[17] N.M. Davies, T.J. Hollowood, V.V. Khoze and M.P. Mattis, *Gluino condensate and magnetic monopoles in supersymmetric gluodynamics*, *Nucl. Phys.* **B 559** (1999) 123 [hep-th/9905015].

[18] P. van Baal, *Instantons versus monopoles*, hep-th/9912035.

[19] N. Dorey, V.V. Khoze and M.P. Mattis, *On mass-deformed $N = 4$ supersymmetric Yang-Mills theory*, *Phys. Lett.* **B 396** (1997) 141 [hep-th/9612231].

[20] R. Donagi and E. Witten, *Supersymmetric Yang-Mills theory and integrable systems*, *Nucl. Phys.* **B 460** (1996) 299 [hep-th/9510101].

[21] E. Witten, *Branes and the dynamics of QCD*, *Nucl. Phys.* **B 507** (1997) 658 [hep-th/9706109].

[22] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B 72** (1974) 461.

[23] V. Balasubramanian, P. Kraus and A. Lawrence, *Bulk vs. boundary dynamics in anti-de Sitter spacetime*, *Phys. Rev.* **D 59** (1999) 046003 [hep-th/9805171].

[24] I.R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking*, *Nucl. Phys.* **B 556** (1999) 89 [hep-th/9905104].

[25] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Holomorphic anomalies in topological field theories*, *Nucl. Phys.* **B 405** (1993) 279 [hep-th/9302103].

[26] E. Witten, *Instantons, the quark model and the* $1/N$ *expansion*, *Nucl. Phys.* **B 149** (1979) 285.

[27] N. Dorey and S. P. Kumar, work in preparation.

[28] A. Gorsky, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, *Integrability and Seiberg-Witten exact solution*, *Phys. Lett.* **B 355** (1995) 466 [hep-th/9505035].

[29] E. Martinec and N. Warner, *Integrable systems and supersymmetric gauge theory*, *Nucl. Phys.* **B 459** (1996) 97 [hep-th/9509161];
E.J. Martinec, *Integrable structures in supersymmetric gauge and string theory*, *Phys. Lett.* **B 367** (1996) 91 [hep-th/9510204].